Ising and Related Models

The Ising model, $H = -\sum_{ij} J_{ij} S_i S_j$, where i,j are vertices of some graph (e.g. a lattice), $S_i = \pm 1$, J_{ij} are real numbers, is the simplest model one could write down. It has a $\mathbb{Z}_2$ ("Ising") symmetry, i.e., it is invariant under $S_i \to -S_i$ $\forall i$. We studied this model in 210A a fair bit and we will here assume the knowledge of the material covered in that course.

Generalizations of Ising model:

(i) $O(N)$ vector model:

$$H = -\sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

where $\vec{S}_i$ is an N-component vector. Note that N is independent of the spatial dimension in which the vectors ("spins") are embedded. The symmetry is $O(N)$:

$\vec{S} = O \vec{S}$ where O is an $O(N)$ matrix, i.e.

$S_\alpha = O_{\alpha\beta} S_\beta$.

(ii) Matrix models:

$$H = \sum_{ij} J_{ij} \operatorname{Tr}[U^\dagger_i U_j]$$ where U is an unitary or orthogonal matrix [i.e. $U^\dagger U = \mathbb{1}$ or $U^T U = \mathbb{1}$]. There are two independent symmetries: left and right multiplication of U:

$U \to V_L U$, and $U \to U V_R$.

(iii) Potts model:

$$H = -\sum_{ij} J_{ij} \delta_{\sigma_i, \sigma_j}$$

where each site i contains a variable $\sigma_i \in 1, 2, \ldots N$. The symmetry is S_N, the symmetric group corresponding to permutation of N objects. For $N=2$, this is just the Ising model. For $N>2$, the symmetry group is distinct from $\mathbb{Z}_N$, and unlike $\mathbb{Z}_N$, is non-abelian.

Relation of Physical problems to simple models

(i) SAW and O(N) model (Cardy sec. 9.3)

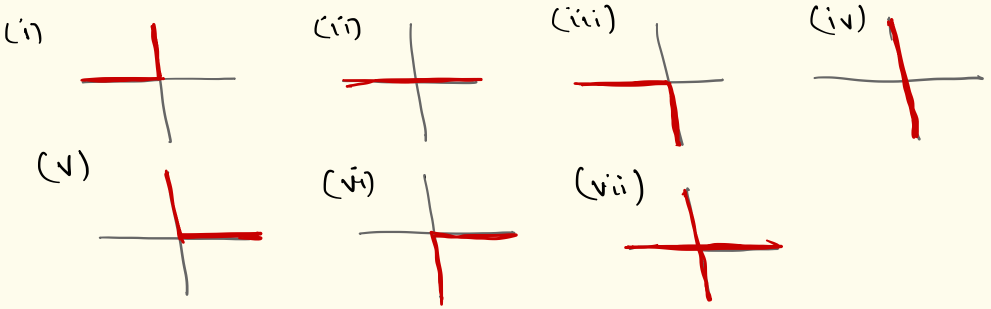
Consider a variation of the O(N) model:

$$H = -\sum_{\langle ij \rangle} \log\left[1 + x\,\vec{S}_i \cdot \vec{S}_j\right]$$

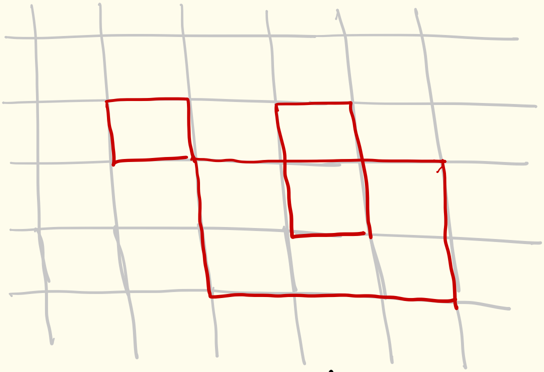
where $\langle ij \rangle$ denotes the nearest neighbor bonds of a d-dim cubic lattice e.g. lets focus on $d=2$ i.e. the square lattice. The partition f^n is

$$Z = \mathrm{Tr}\, e^{-H} = \mathrm{Tr} \prod_{\langle ij \rangle} \left[1 + x\,\vec{S}_i \cdot \vec{S}_j\right]$$

Expanding all terms, one obtains 2^{N_B} terms where N_B denotes total number of bonds. Only those terms survive after trace where each $\vec{S}_i$ appears an even number of times. Thus at a given vertex only the following configs contribute:



Thus, the partition f^n gets contribution only from closed loop configs, where intersection of loops is allowed due to the config. # (vii).



etc.

The contribution from a single closed loop with n_B bonds

is: $x^{n_B} \, Tr[\vec{S}_1 \cdot \vec{S}_2 \; \vec{S}_2 \cdot \vec{S}_3 \; \ldots \; \vec{S}_{n_B - 1} \cdot \vec{S}_1]$

$= x^{n_B} \; Tr[S_{1a} S_{2a} \; S_{2b} S_{3b} \; \ldots \; S_{n_B - 1 \, k} \; S_{1k}]$

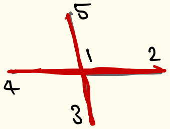
using the normalization $Tr[S_{1a} S_{1k}] = \delta_{ak}$

$\Rightarrow$ The contribution is $x^{n_B} \, n$ where n is the n in $O(n) = \delta_{aa}$.

Thus, $$Z = \sum_{\substack{\text{loop} \\ \text{config.}}} n^{\#\text{ of loops}} \; x^{\#\text{ of bonds occupied by loops}}$$

Taking $n \to 0$, Z gets contribution only from those configs. with only a single loop.

The $n \to 0$ limit also suppresses the configs: (viii)



since in $\mathrm{Tr}\{(S_1 \cdot S_2)(S_1 \cdot S_3)(S_1 \cdot S_4)(S_1 \cdot S_5) \cdots\}$, the trace over S_1 leads to a factor $\mathrm{Tr}[S_1^a S_1^b S_1^c S_1^d]$

which, due to $O(n)$ sym. is proportional to $\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$. The proportionality factor must be some f^n of n. Let's write $\mathrm{Tr}\{S_1^a S_1^b S_1^c S_1^d\}$

$= A(n)[\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}]$. Setting $a=b$, $c=d$ and summing over a, c, the LHS goes

as $\sum_{ac} \mathrm{Tr}[S_1^a S_1^a S_1^c S_1^c] = \delta_{aa}\delta_{cc} = n^2$.

The RHS goes as $A(n)[\delta_{aa}\delta_{cc} + 2\delta_{ac}\delta_{ac}]$

$$= A(n)[n^2 + 2n] = A(n)\, n(n+2).$$

$$\Rightarrow A(n) = \frac{n}{n+2} \to 0 \text{ as } n \to 0.$$ Therefore,

the above config., which corresponds to intersecting closed loops, does not contribute to the partition f^n Z as $n \to 0$. Using similar argument one can show that the correlation f^n $\lim_{n\to 0} \dfrac{\mathrm{Tr}\, S(\vec{r})\, S(\vec{r}')\, e^{-H}}{Z}$

equals the grand-canonical partition f^n $Z(x)$ defined earlier:

$$Z(x) = \sum_N M_N(\vec{R})\, x^N$$ where the sum is over all SAWs from $\vec{r}$ to $\vec{r}'$, $\vec{R} = \vec{r} - \vec{r}'$, N is the length of a given SAW and $M_N(\vec{R})$ is the number of walks of length N. See Cardy Sec. 9.3 for more details. We might revisit this connection later.

(ii) Magnetism and Ising-like models:

This relation is rather direct: magnetic systems are indeed described by Ising or $O(N)$ models — the Ising spins or the $O(N)$ vectors correspond to the electronic spin or orbital angular variable. Since electronic spin is intrinsically quantum mechanical, we will also study quantum Ising model below, and its connection to the classical Ising model.

(iii) Superfluidity and $O(2)$ model:

The order parameter for superfluidity is $\langle b \rangle$ where b is the boson creation operator. This is a complex number. Writing $\langle b_r \rangle = e^{i\varphi_r}$, a lattice model that describes finite-T superfluidity to normal phase transition is an $O(2)$ model: $H = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$.